

Filter Out High Frequency Noise in EEG Data Using The Method of Maximum Entropy

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Abstract. We propose a maximum entropy (ME) based approach to smooth noise not only in data but also to noise amplified by second order derivative calculation of the data especially for electroencephalography (EEG) studies. The approach includes two steps, applying method of ME to generate a family of filters and minimizing noise variance after applying these filters on data selects the preferred one within the family. We examine performance of the ME filter through frequency and noise variance analysis and compare it with other well known filters developed in the EEG studies. The results show the ME filters to outperform others. Although we only demonstrate a filter design especially for second order derivative of EEG data, these studies still shed an informatic approach of systematically designing a filter for specific purposes.

Keywords: EEG, Current source density, Non-recursive filter, Maximum entropy

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INTRODUCTION

Filtering out high frequency noise becomes extremely crucial in source localization problem of invasive EEG studies. Based on Freeman and Nicholson's works, source localization requires calculation of second order derivative of the EEG data [1, 2]. Because the derivative amplifies especially high frequency noise, Freeman and Nicholson proposed to introduce low-pass non-recursive filters to smooth it. [2]. Suppose a set of discrete data, $\Phi(r)$ labeled by spatial position r , which are distorted by noises ε_r and these data are equally spaced sampled with interval h . The simplest non-recursive filter is then defined by simply averaging data at r and its neighboring data with specific set of weighting values p_l ,

$$\Phi_s(r) = \sum_{l=-z}^z p_l \Phi(r + lh) , \quad (1)$$

where z is an integer and $2z + 1$ data points are considered. We will designate this $2z + 1$ -point spatial filter as a $S_{(2z+1)}$ filter. The mean filter, in which p_l equals to $1/(2z+1)$, is one filter widely used in many fields for data smoothing. Its applicability, however, is sometimes limited to data types [3].

Many methods have been developed to design low-pass filters in many fields. For example, there are adaptive beam-forming ([4] - [7]), wavelet de-noising [8], and entropic method ([9] - [14]) etc.. Yet no explicit discussions are given on smoothing noise in second order derivative of contaminated EEG data since Freeman and Nicholson's pioneering works. Freeman and Nicholson's approach will be briefly discussed in Sec. 2.

Two drawbacks, requirements of some empirical guidelines and low accuracy of second order derivative, in their approach will also be addressed.

Our goal is to provide a ME based approach to filter out high frequency noise especially in second order derivative of contaminated EEG data. This approach eliminates the two drawbacks in Freeman and Nicholson's approach and provides a systematic and robust method for smoothing noise. The ME approach and its application in the EEG studies will be presented in Sec. 3 and 4 respectively. We thereafter examine the proposed filter's performance and compare to some other filters such as the mean filter, Freeman and Nicholson's [2], and Rappelsberger's [15] filters through frequency and noise variance analysis in Sec. 5. A conclusion is given at last.

SOURCE LOCALIZATION FROM THE INVASIVE EEG DATA

In neurophysiology, electric field potentials are initiated during neural activity when ions flow across localized regions of cell membranes. These ionic flows establish a distribution of current sources and sinks in the extracellular space [1, 2, 16]. These field potentials can be measured in extracellular space discretely by the invasive EEG instrument, which usually has 16 to 32 detectors on its probe.

According to theory of electromagnetism, the current sources and sinks are associated with second order derivative of the field potentials. Freeman and Nicholson proposed the current source density (CSD) method for source localization [1, 2]. The CSD computes second order derivative approximately through substituting specific smoothed $\Phi_s(r)$, Eq. (1), into $Ta_3(r) = \sigma_c/h^2 (\Phi(r-h) - 2\Phi(r) + \Phi(r+h))$, where three data are considered, σ_c is conductivity, and h is separating distance between two sensors. A final expression of an approximated second order derivative is given by

$$D_2(r) = \sigma_c/h^2 \sum_{m=-M}^M a_m \Phi(r+mh), \quad (2)$$

from $2M+1$ datum with weightings $a_{\pm M} = p_z$, $a_{\pm M \mp 1} = p_{z-1} - 2p_z$, and $a_{\pm M \mp k} = p_{z-k} - 2p_{z-k+1} + p_{z-k+2}$, in which $k \geq 2$. The preferred smoothing function should be able to reduce high frequency noise and noise amplified by operation of the $Ta_3(r)$. Freeman and Nicholson devised several smoothing functions as shown in Table. 1, denoted by FNSn ([2]), where Sn stands for n-point spatial smoothing. Notes that the FNS₃(2) is still used in Karwoski et al. [17] and Ulbert et al. [18]. These p_l are derived from the least square method [19, 20]. In addition, they also introduce some ad hoc rules to obtain 5 and 7-point filters. For example, the FNS₅ is obtained by smoothing data twice using set of (3/10, 4/10, 3/10). Later, several successors have designed other smoothing functions according to their needs such as the Rappelsberger's RS₃ filter [15].

There are two drawbacks in Freeman and Nicholson's CSD method. The first is requirements of some ad hoc empirical guidelines for designing smoothing functions such as FNS₃(1) or RS₃ shown in Table. 1. These empirical guidelines may limit the use of these smoothing functions to some specific EEG data. For example, Freeman and Nicholson's smoothing functions are suited for Anuran cerebellum studies [2]. The second is these smoothing functions are devised for second order derivative calculated via

$Ta_3(r)$, which is not an accurate calculation. In next section, we propose an informatic approach based on the ME method to eliminate these two drawbacks.

TABLE 1. Filters in EEG studies. The table lists five smoothing functions [2, 15]. MF: mean filter, FN: Freeman and Nicholson, R: Rappelsberger.

	p_0	$p_{\pm 1}$	$p_{\pm 2}$	$p_{\pm 3}$
MFS ₃	0.33	0.33	0	0
FNS ₃ (1)	0.5	0.25	0	0
FNS ₃ (2)	0.43	0.29	0	0
RS ₃	0.54	0.23	0	0
FNS5	0.34	0.24	0.09	0
FNS7	0.26	0.21	0.12	0.04

A MAXIMUM ENTROPY LOW-PASS FILTER

Rationale. The method of ME is developed as a tool for assigning a probability distribution of observing a system to be at a certain state according to information in hand, which is in the form of constraints [21]. Based on conventional studies on properties of the noise filter, Eq. (1), in the EEG source localization problems, which will be illustrated later, one can treat the weightings p_l as a probability distribution. Thus the ME provides a robust and objective approach for determining a preferred set of p_l once constraints relevant to noise reduction in the EEG problems are given.

The constraints. Conventional studies in EEG data analysis have shown some properties that are relevant to smooth noise using Eq. (1). First, the p_l are positive decimal values and symmetric around $l = 0$. These values obey a normalization condition $\sum_{l=-z}^z p_l = 1$. Second, the preferred set of the p_l should have noise in data being mostly removed through the $\Phi_s(r)$. We expand right hand side of Eq.(1) with respect to r according to the Taylor expansion and use $\sum_{l=-z}^z p_l = 1$ and $p_l = p_{-l}$, which results in the odd order derivative terms in the expansion vanished. The final result is $\Phi_s(r) = \Phi(r) + \Phi^{(2)}(r) h^2 \delta_2 / 2 + \dots$, where $\Phi^{(2)}(r)$ denotes exact second order derivative of clean data and so on and $\delta_2 = \sum_{l=-z}^z p_l l^2$. It indicates that smoothing ability is actually related to the δ_2 value. Summarizing the above results, suggests the weightings p_l to be a probability distribution of having the Eq.(1) to reduce noise by the δ_2 approximately. Thus the ME can be applied to determine the preferred set of p_l .

The preferred low-pass filter for data smoothing. The ME states that maximizing entropy, $S = -\sum_{l=-z}^z p_l \ln p_l$, subject to the two constraints, the normalization condition of p_l and $\delta_2 = \sum_{l=-z}^z p_l l^2$ gives a preferred form of the p_l ,

$$p_l = Z^{-1} \exp(-\alpha l^2), \quad (3)$$

where α is a Lagrangian multiplier and partition function $Z = \sum_{l=-z}^z \exp(-\alpha l^2)$. The preferred set of p_l is a function of α . Namely, the ME method generates a family of $p_l(\alpha)$ distribution. In principle, the Lagrangian multiplier α can be determined by substituting Eq. (3) back into $\delta_2 = \sum_{l=-z}^z p_l l^2$, which gives $\delta_2 = -Z^{-1} \partial Z / \partial \alpha$, if the δ_2

value is given. Unfortunately, the δ_2 value is unknown at this point. Thus, we propose an alternative method to determine the α .

Since the preferred α value should result in corresponding set of the p_l to mostly reduce noise, if one can quantify noise the preferred α value can be determined. Although it is difficult to quantify noise, noise variance can be quantified by the autocorrelation function [3, 6]. Suppose field potentials $\Phi(r) = \Phi_0(r) + \varepsilon_r$ is repeatedly measured several times, in which a Gaussian white noise ε_r with variance σ is considered. For white noise, we have $\langle \varepsilon_r \rangle = 0$. Besides, noises are uncorrelated at different data points r and r' , $\langle \varepsilon_r \varepsilon_{r'} \rangle = \sigma^2$ for $r = r'$ and 0 for $r \neq r'$ [3]. Because noise has zero mean, mean $\langle \Phi_0(r) \rangle$ is identical to $\Phi_0(r)$, noise-free data. Thus one can find autocorrelation of $\Phi(r)$ is $C_o = \langle (\Phi(r) - \langle \Phi(r) \rangle)^2 \rangle = \langle (\varepsilon_r - \langle \varepsilon_r \rangle)^2 \rangle = \sigma^2$, the noise variance. Similarly, one can compute the autocorrelation function of $\Phi_s(r)$, $C_{\text{smooth}} = \langle (\Phi_s(r) - \langle \Phi_s(r) \rangle)^2 \rangle = \sigma^2 \sum_{l=-z}^z p_l^2$. It indicates noise variance being amplified by $\sum_{l=-z}^z p_l^2$ after data is processed by the Eq.(1). Thus the preferred α value should minimize the noise variance C_{smooth} , which is found to be zero and $p_l(\alpha = 0) = 1/(2z + 1)$. This is exactly the mean filter obtained from the least square method [3, 19]. Yet the ME approach roots in reducing noise variance C_{smooth} mostly, which is not clear in the least square method.

FILTER OUT NOISE IN SOURCE LOCALIZATION

Basic features. Freeman and Nicholson suggest to identify current sources and sinks through calculating second order derivative of the field potential via the $\text{Ta}_3(r)$ with specific smoothed $\Phi_s(r)$. However, the $\text{Ta}_3(r)$ does not provide accurate calculation. One straightforward way to raise the accuracy is to incorporate more data points into for calculation,

$$\text{Ta}_{2M+1}(r) = \sum_{m=-M}^M a_m^T \Phi(r + mh), \quad (4)$$

where $2M+1$ data points are considered. To determine coefficients a_m^T , one first expand $\Phi(r + mh)$ with respect to r based on the Taylor expansion, $\text{Ta}_{2M+1}(r) = \sum_{m=-M}^M a_m^T \Phi(r) + \sum_{m=-M}^M a_m^T mh \Phi^{(1)}(r) + \sum_{m=-M}^M a_m^T (mh)^2 / 2 \Phi^{(2)}(r) + \dots$. By requesting coefficient of second order term, $\sum_{m=-M}^M a_m^T (mh)^2 / 2 = 1$, and rests are zero, we have approximated $\text{Ta}_{2M+1}(r)$ to be identical to exact second order derivative $\Phi^{(2)}(r)$. One then can solve for coefficients a_m^T given these criteria. For three point calculation, coefficients are given in the Ta_3 . Coefficients $a_{\pm 2}^T = -1/12$, $a_{\pm 1}^T = 16/12$, and $a_0^T = -30/12$ for Ta_5 and $a_{\pm 3}^T = 1/90$, $a_{\pm 2}^T = -13.5/90$, $a_{\pm 1}^T = 135/90$, and $a_0^T = -245/90$ for Ta_7 .

Given these coefficients, one can immediately find the noise variance to be amplified by $C_{D_2} = \langle (D_2(r) - \langle D_2(r) \rangle)^2 \rangle = \sigma^2 \sum_{m=-M}^M (a_m^T)^2$ after operation of the $\text{Ta}_{2M+1}(r)$. For example, it is 6, 8.03, 9.75 times of the original noise variance for using $\text{Ta}_3(r)$, $\text{Ta}_5(r)$, and $\text{Ta}_7(r)$ respectively. The noise amplification is proportional to the number of data utilized in the calculation. This result suggests a trade-off of raising the accuracy of

TABLE 2. The preferred ME filters, denoted by MES_nTa_m .

	$C_{D_{s2}} (\times \sigma^2)$	α	p_0	$p_{\pm 1}$	$p_{\pm 2}$	$p_{\pm 3}$
MES_3Ta_3	0.28	0.031	0.42	0.28	0	0
MES_3Ta_5	0.38	0.033	0.43	0.28	0	0
MES_3Ta_7	0.42	0.035	0.44	0.27	0	0
MES_5Ta_3	0.048	0.017	0.29	0.23	0.12	0
MES_5Ta_5	0.061	0.018	0.29	0.23	0.11	0
MES_5Ta_7	0.065	0.019	0.30	0.23	0.11	0
MES_7Ta_3	0.014	0.011	0.22	0.19	0.12	0.06
MES_7Ta_5	0.017	0.012	0.23	0.2	0.12	0.05
MES_7Ta_7	0.019	0.012	0.23	0.2	0.12	0.05

second order derivative calculation with more data is amplification of the noise variance. Thus it is unlikely that the Freeman and Nicholson's and others filters can smooth this noise amplification because those filters are designed for $\text{Ta}_3(r)$ calculation only. One needs new smoothing functions.

The preferred ME filter. The new smoothing functions can be obtained through the following procedure. Substituting smoothed data, Eq. (1) into second order derivative $\text{Ta}_{2M+1}(r)$ first gives a generic expression

$$D_{s2}(r) = \sum_{n=-(M+z)}^{M+z} a'_n \Phi(r + nh), \quad (5)$$

where a'_n is a function of a_m^T and p_l . The preferred set of p_l is then determined by minimizing variance $C_{D_{s2}} = \langle (D_{s2}(r) - \langle D_{s2}(r) \rangle)^2 \rangle = \sigma^2 \sum_{n=-(M+z)}^{M+z} a_n'^2$. It should have high accuracy of second order derivative calculation and minimum noise variance simultaneously. Table 2 gives the results. We define operation based on the ME that utilizes n data points for noise smoothing, S_n , and m data for calculating second order derivative, Ta_m , as MES_nTa_m in the first column of Table. 2. Second column records the minimum $C_{D_{s2}}$ value. Next five columns list the corresponding α and the p_l value consecutively. The table first shows that when there is more data, S_3 , S_5 , and S_7 , to be considered for smoothing given Ta_m , the noise variance $C_{D_{s2}}$ will be decreased dramatically. However, when one smoothes data with S_n and calculate second order derivative using more number of smoothed data points, Ta_3 , Ta_5 , and Ta_7 , the noise variance $C_{D_{s2}}$ will then be amplified as expected.

PERFORMANCE ANALYSES

Instead of applying the proposed ME filter directly to study real EEG data for examinations, we will only analyze the filters's performance through frequency and noise variance analysis in this work. In addition, we will compare it to other filters, mean filter (MF), Freeman and Nicholson's (FN), Rappelsberger's (R) filters from Table 1.

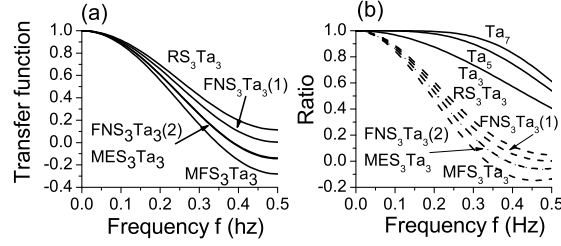


FIGURE 1. Frequency analysis of the five different 3-point S_3 filters in panel (a) and different S_3Ta_3 in panel (b).

Frequency analysis. The frequency analysis applies the transfer function $H(\omega)$ for examining changes of signal's amplitudes after signal being processed by the filters and second order derivative calculation [3]. The transfer function $H(\omega) = \exp(-i\omega r)\Phi_s(r) = p_0 + 2\sum_{l=1}^Z p_l \cos l\omega$ is derived by substituting $\Phi(r) = \exp(i\omega r)$, which is shown to be an eigenfunction of a linear time-invariant system, into $\Phi_s(r)$ of the Eq.(5), where angular frequency ω equals to $2\pi f$ rotational frequency [3]. When the transfer function is less than one, it indicates signal amplitude being attenuated by the filter. Panel (a) of Fig. 1 plots the results of using the different filters from Table 1. This figure demonstrates the mean filter has the largest amplitude attenuation and is followed by $FNS_3(2)$ and MES_3 , $FNS_3(1)$, and RS_3 consecutively.

Similarly, one can compute the transfer function of second order derivative of smoothed data. Afterward, we compute the ratio of calculated and real second order derivative value through $R(\omega) = -D_{s2}(r)/\omega^2 = -1/\omega^2 (a_0 + 2\sum_{l=1}^{M+z} a'_l \cos l\omega)$ for explicit comparison of effects of the smoothing. When the ratio $R(\omega)$ equals to one, the calculated second order derivative of the smoothed data is identical to real value. Panel (b) of Fig. 1 plots not only the calculation through five filters, but also the three second order derivative calculations Ta_3 , Ta_5 , and Ta_7 respectively. First of all, frequency attenuation is the smallest in Ta_7 , and is followed by Ta_5 and Ta_3 . That explains again, why the second order derivative is more accurate when more data is comprehended for calculation. Given the second order derivative calculation with fixed number of smoothed data, for example, all S_3Ta_3 in this figure, it suggests that the mean filter MFS_3Ta_3 again has signal with high frequency attenuated the most among these filters. These attenuation patterns are similar to the panel (a), which is not surprising because smoothing data first and then calculating second order derivative is identical to calculating second order derivative first and then smooth the calculation.

Noise variance analysis. It seems the mean filter to be superior to all other choices from the previous analysis. However, the following noise variance analysis shows different results. We substitute the filters from Table 1 into Eq. (4) to compute new coefficient a'_m , and evaluate noise variance $C_{D_{s2}}$. The results are listed in Table 3. Because we design the ME filter to mostly reduce noise variance, the analysis shows the MES_3Ta_3 filter to exactly has the best performance among all filters.

Besides, Freeman and Nicholson have found noise in field potential measurements to be proportional to a K-value, $K = \sum_{l=-(M+z)}^{M+z} |a'_l|$ [2]. Thus we calculate the K-value

TABLE 3. Comparisons of filters. The table lists a'_n value and noise variance $C_{D_{s2}} (\times \sigma^2)$ in the third column and K value derived by Freeman and Nicholson for noise estimate in the fourth column.

	a'_0	$a'_{\pm 1}$	$a'_{\pm 2}$	$a'_{\pm 3}$	$a'_{\pm 4}$	$C_{D_{s2}} (\times \sigma^2)$	K
MFS ₃ Ta ₃	0	-0.333	0.333	0	0	0.44	1.33
FNS ₃ Ta ₃ (1)	-0.5	0	0.25	0	0	0.38	1.01
FNS ₃ Ta ₃ (2)	-0.286	0.143	0.286	0	0	0.28	1.13
RS ₃ Ta ₃	-0.623	0.082	0.229	0	0	0.57	1.34
MES ₃ Ta ₃	-0.284	-0.144	0.286	0	0	0.28	1.14
FNS ₅ Ta ₃	-0.2	-0.05	0.06	0.09	0	0.073	0.63
FNS ₇ Ta ₃	-0.1	-0.04	0.01	0.04	0.04	0.019	0.39
MES ₅ Ta ₃	-0.116	-0.055	-0.007	0.12	0	0.048	0.48
MES ₇ Ta ₃	-0.06	-0.038	0.003	0.003	-0.06	0.014	0.27

after using different filters and list in the fourth column of Table 3. The K-value shows although MES₃Ta₃ is slightly outperformed by the FNS₃Ta₃(1), which has the same performance as the FNS₃Ta₃(2), the ME filter is superior than all other filters when more data points are included for smoothing.

Discussions. The frequency analysis has shown the mean filter MFS₃ and the MES₃ and FNS₃(2) mostly attenuate high frequency part in the signal. Yet the noise variance analysis, comparison of the $C_{D_{s2}}$ and K value, shows the MES₃Ta₃ and the FNS₃Ta₃(2) can mostly reduce noise variance among all three-point filters. These three analyses suggests the ME filter not only mostly reduce noise variance in second order derivative but also has the ability to mostly attenuate high frequency signal.

CONCLUSIONS

The filter design delicately depends on one's purpose and properties of a target system. It sometimes involves some ad hoc empirical guidelines that suit ones need. For example, Freeman and Nicholson have designed several filters for their needs [2]. We propose and demonstrate an informatic approach to design a low-pass filter based on the method of ME. In addition, we examine and compare the preferred ME filter with other well known filters in EEG studies through frequency and noise variance analysis. The performance of the preferred ME filter MES₃Ta₃ and the FNS₃Ta₃(2) filter are found to be almost identical. Even though the ME filter is designed for second order derivative calculation especially for the EEG data in this work only, the approach's systematics and flexibility ensures one can design filters that suit other purposes robustly without introducing ad hoc empirical guidelines.

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