

A NEW FAMILY OF N-STATE REPRESENTATIONS OF THE BRAID GROUP

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Abstract

For the Artin braid group B_n , we give two types of N-state representations for $N=2,3,4,5,6$.

1. Introduction

The theory of braids and knots is related to many topics in physics. Recently a connection between knot theory and the theory of exactly solvable models in statistical mechanics and many body systems in two dimensions has been discovered [1-5]. Central to this is the Yang-Baxter relation which is a sufficiency condition for the transfer matrices to commute in those models. Exploiting the similarity between the defining relation of Artin's braid group B_n and the Yang-Baxter relation, various representations of B_n have been obtained from statistical models at criticality. The Markov trace defined on B_n is used to construct topological invariants for knots and links. State models associated to link diagrams have been shown to be partition functions of certain statistical models. For instance, Kauffman [6] has defined a state model for the Alexander-Conway polynomial which can be seen as the low temperature limit of the partition function of a generalized Potts model. Methods for constructing link polynomials via the braid group or a diagrammatic approach rely on a two-dimensional definition of link polynomials. Recently within the context of a solvable topological quantum field theory in 2+1 dimensions, Witten [7] has proposed an intrinsically three-dimensional definition of the Jones polynomials. This idea has been carried further and a new hierarchy of link polynomials has been derived from a topological Chern-Simons gauge theory [8]. The braid group is of interest in its own right, as the fun-

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damental group of the configuration space of n indistinguishable particles in two dimensions. Braid group representations are related to strange statistics (neither Bose or Fermi) in quantum mechanics in 2+1 dimensions [9-12] and quantum field theory in 1+1 dimensions [13] and to the monodromy of multipoint correlation functions in two dimensional conformal field theory [14,15].

In an earlier paper [12], we proposed an algorithm for constructing N -state representations of B_n of the maximally symmetric type (defined in section 2); from this family of representations one can derive an infinite sequence of link polynomials. The $N=2$ case in this family corresponds to the famous Jones polynomial. The method consisted of solving a set of equations of the Yang-Baxter type. The resulting family of solutions had been first discovered by Akutsu and Wadati [1,2]; their method consisted of extracting representations of B_n from exactly solvable N -state vertex models in statistical mechanics at criticality. Recently our method led to the discovery of a new family of N -state representations of B_n ; solutions for $N=2, 3$ and 4 were given [16,17].

The object of my talk (M.C.) is to give a detailed account of those representations. In section 2 solutions to both families of representations are given up to $N=6$.

2. N-State Representations of B_n

Artin's braid group B_n [18,19] is generated by a set of $(n-1)$ generators (elementary braids) $g_1, g_2, \dots, g_{n-1}$ and their inverses subject to the following necessary and sufficient defining relations

$$g_i g_j = g_j g_i \quad |i-j| \geq 2 \quad (2.1a)$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \quad (2.1b)$$

An element β in B_n , called a braid, is a word in the g_i 's

$$\beta = g_{i_1}^{\sigma_1} g_{i_2}^{\sigma_2} \dots, \quad \sigma_i \in \mathbb{Z}$$

Let V be an N -dimensional vector space and $R \in \text{End}(V \otimes V)$ be an $N^2 \times N^2$ matrix that has an inverse. The mapping

$$\rho : B_n \rightarrow \text{End}(V^{\otimes n}) \quad (2.2)$$

with

$$\rho(g_i) = I_1 \otimes \dots \otimes I_{i-1} \otimes R \otimes I_{i+2} \otimes \dots \otimes I_n \quad (2.3)$$

is a representation of B_n , where $I \in \text{End}(V)$ is the identity matrix, the subscript i means the i^{th} vector space in $V^{\otimes n}$, and R acts on the i^{th} and $(i+1)^{\text{th}}$ vector spaces. The form of (2.3) insures that (2.1a) holds. Let us now consider (2.1b). Component wise :